

# Instanton solutions in the problem of wrinkled flame fronts dynamics.

Dmitri Volchenkov <sup>1,2</sup> and Ricardo Lima <sup>1</sup>

February 9, 2008

<sup>1</sup> *Centre de Physique Theorique, 13288 Marseille Cedex 09, Luminy Case 907, France*

<sup>2</sup> *Institute of Physics, Saint-Petersburg State University, Peterhof, Ulyanovskaya 1, 198904, Russia*  
*volchen@cptsg2.univ-mrs.fr, lima@cpt.univ-mrs.fr*

PACS number(s): 47.10.+g, 47.27.-i, 82.40.Py, 47.54.+r, 05.40.+j,

## Abstract

The statistics of wrinkling flame front is investigated by the quantum field theory methods. We dwell on the WKB approximation in the functional integral which is analogous to the Wyld functional integral in turbulence. The main contribution to statistics is due to a coupled field-force configuration. This configuration is related to a kink between metastable exact pole solutions of the Syvashinsky equation. These kinks are responsible for both the formation of new cusps and the rapid power-law acceleration of the mean flame front. The problem of asymptotic stability of the solutions is discussed.

## 1 Introduction

It had been shown in [1] that under a weakly nonlinear approximation, the dynamics of a wrinkled planar flame front is governed by a nonlinear partial differential equation (PDE)

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2} U_b \left( \frac{\partial \Phi}{\partial x} \right)^2 + D_M \frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{2} \gamma U_b \Lambda \{ \Phi \}. \quad (1)$$

Here,  $\Phi$  is the interface of a distorted planar flame,  $U_b$  is the speed of the planar flame relative to the burning gas,  $D_M$  is the Markstein diffusivity and  $\gamma$  is the thermal expansion coefficient,

$$\gamma = \frac{(\rho_f - \rho_b)}{\rho_f}, \quad (2)$$

where  $\rho_f$  is the density of the fresh mixture and  $\rho_b$  is the density of the burned gas,  $\rho_f > \rho_b$ . The equation (1) is asymptotically exact in the limit of small  $\gamma \ll 1$ .  $\Lambda \{ \dots \}$  represents a linear

singular nonlocal operator defined conveniently in terms of the spatial Fourier transform by:

$$\Lambda : \tilde{\Phi}(k, t) \longmapsto 2\pi|k|\tilde{\Phi}(k, t), \quad \tilde{\Phi}(k, t) = \int_{-\infty}^{+\infty} dk \Phi(x, t)e^{2\pi i k x}. \quad (3)$$

$\Lambda$  is responsible for the Darrieus-Landau instability [2, 3].

Direct numerical simulations for (1) performed in [4] show that even when the initial conditions is chosen to be smooth, the cusps develop on the flame interface as time increases. When the integration domain is wide enough, the secondary randomlike subwrinkles arise on the interface. This wrinkling process is accompanied by the flame speed enhancement undergoing an acceleration in time [7], namely the mean radius seems to increase with time according to a  $t^{3/2}$  power-law.

Numerous analytical investigations devoted to (1) display that in the limit of long times the local flame dynamics is driven by the large-scale geometry [8]-[10]. Exact solutions of (1) can be obtained in principle by using the pole decomposition technique [7],[11]- [13]. For such pole solutions, (1) formally reduces to a finite set of ordinary differential equations (ODE) which describe the motion of the poles in the complex plane. These poles are interpreted as to be related to the cusps observed in physical space. However, numerical and analytical results demonstrate convincingly that the solutions of the ODEs do not resemble those obtained from the direct numerical integration of (1), [7]. In particular, the number of wrinkles obtained from the ODEs is independent of time (see [11]) and the corresponding (mean) expansion of the front is much slower than the  $t^{3/2}$  power law.

In [5]-[7], it was argued that the inconsistencies with the pole decomposition method lie in the stability of the exact pole solutions. The initial value problem of the linearised PDE about a pole solution has been solved numerically, as a result they concluded that pole solutions are unstable for large  $\gamma$ . Consequently, they are not observed in experiments.

It was conjectured in [15] that nonlinearity alone is not enough to meet the experimental observations and that the results of the spectral numerical integrations is due to computational noise. In [15] a model had been developed, where pseudorandom forcing is included. It is shown that many broad-banded exciting fields indeed lead to the rapid spawning of wrinkles.

Analyzing the Fourier spectrum of the solution of PDE obtained numerically in [7], one observes that it undergoes the local kinks at very short time. Occurring in the amplitude of Fourier modes and its harmonics, these kinks, are followed by an oscillation of this amplitude which corresponds, in physical space, to the birth of new pairs of wrinkles.

The linear stability of the pole decomposition solutions was discussed in [14] in details. The exact analytical expressions for the eigenvalues and eigenfunctions have been constructed. Based on these expressions, in [14], they demonstrate that for any value of the parameter  $\gamma$  there exists the only asymptotically stable solution with the largest possible (for this particular value of  $\gamma$ ) number of poles,  $N_\gamma$ . As the parameter  $\gamma$  increases, the equilibrium states of the PDE undergo a cascade of bifurcations. In this way the new solution with  $N$  poles gains stability

while the former one with  $N - 1$  poles becomes unstable. However, the nonlinear stability and dynamics of cusps still remain an important open question within such an approach, [14].

In the present paper we continue the investigation of the kink type behavior of the expanded flame front which has been started in [5]-[7]. In contrast to the study in [14], we keep the value of  $\gamma$  fixed and small and use a pseudorandom force as an origin of the spawning of wrinkles.

We demonstrate that the main contribution to the statistics of wrinkling flame front is given by a coupled field-force configurations - the *instantons*. These configurations are related directly to a very short time (practically instant) kinks between metastable *groundstates* incident to different numbers of poles. We should stress that we address the stochastic problem based on the Syvashinsky equation (1), but not the exact solutions of this equation.

The paper is organized as follows. In Sec. II, we formulate the stochastic problem for the equation governing the slope function dynamics of advancing flame fronts. These fronts usually either form fractal objects with contorted and ramified appearance or they wrinkle producing self-affine fractals characterized with some critical exponent, [16]. In Sec. III, we analyse the problem from the point of view of critical phenomena theory. We demonstrate that the  $t^{3/2}$  power law observed experimentally would be a direct consequence of the general Kolmogorov's scaling with the critical dimensions of time  $\Delta_t^K = -2/3$  and velocity  $\Delta_v^K = -1/3$  (the dimension of  $x$  is taken by definition as  $\Delta_x = -1$ ) which are well known in the fully developed turbulence theory (see [18] for a review).

We must say that in actual problem the renormalization group technique (RG) (which has proved itself so well in the fully developed turbulence theory, [18], [19]) is ultimately ineffective since, obviously, the regime of critical scaling is not attained. One even can hardly use the concept of critical dimensions for the actual quantities.

The examples of successful application of the saddle point calculations to the Burger's equation [26] and to the description of intermittency phenomenon in turbulence [27] have been given recently. These papers have inspired us to employ this technique in the problem of wrinkling flame front expansion. The infinite set of instanton like solutions which we have found is dramatically dissimilar to those computed in [26] and [27].

In Sec. IV, we construct the statistical theory of wrinkles based on the action functional relevant to the actual stochastic problem.

The minimization of action discussed in Sec. V requires that the field and force to be coupled in some particular configurations. We also illustrate the instanton mechanism of poles generation for the particular two poles initial configuration. The process keeps repeating itself as time increases. We then conclude in the last section.

## 2 The stochastic problem for the Syvashinsky equation

The stochastic problem for the equation governing the slope function dynamics of the flame front,  $u(x, t) = \partial_x \Phi(x, t)$ , with a Gaussian distributed pseudorandom force included in the r.h.s. reads as follows

$$\partial_t u + U_b u \partial_x u = D_M \partial_x^2 u + \frac{1}{2} \gamma U_b \Lambda \{u\} + f. \quad (4)$$

The pair correlation function for  $f$  is taken in the form

$$\langle f(x, t) f(x', t') \rangle = D_f(x - x') \delta(t - t'), \quad (5)$$

in which the function  $D_f(x - x')$  is supposed to be an even smooth "bell"-shaped function of  $x$ . To be specific, we take it in the form

$$D_f(x) = \frac{D_0}{\pi} \frac{m}{x^2 + m^2} \quad (6)$$

decaying at the rate  $m$  and turning into  $D_0 \delta(x)$  as  $m \rightarrow 0$ , where  $D_0$  is a constant.

The equation (4) is similar to the Burger's equation except the singular term,  $\propto \gamma U_b \Lambda \{u\}$ . The homogeneous unforced equation (4) considered in [11] in details. In particular, it was shown that it possesses a pole decomposition, i.e., it allows a countable number of uniform solutions,

$$u(x, t) = -2\nu \sum_{i=-N}^N \frac{1}{x - z_i(t)}, \quad (7)$$

in which  $z_i$ 's are poles in the complex plane (coming in complex conjugate pairs) moving according to the laws of motion of poles

$$\dot{z}_i = -2\nu \sum_{i \neq j} \frac{1}{z_i - z_j} - i\gamma U_b \operatorname{sign} [\Im(z_i)], \quad (8)$$

where  $\Im$  denotes the imaginary part of a pole. One can derive easily the corresponding steady ( $\dot{z}_i = 0$ ), solution of the Sivashinsky equation for the simplest configurations concerning the minimal number of poles. For example, for two poles the only steady solution is given by

$$u^{(2)}(x) = -\frac{4D_M x}{x^2 + D_M^2}, \quad (9)$$

and there are two possible four-pole steady configurations,

$$u^{(4)}(x) = \pm \frac{4D_M(\pm 2x^3 + 27\sqrt{2}iD_M^3 + 9\sqrt{2}iD_M x^2)}{-x^4 \pm 54\sqrt{2}iD_M^3 x \pm 6\sqrt{2}iD_M x^3 + 81D_M^4}. \quad (10)$$

In [5]-[7], by numerically solving the initial value problem that results from linearizing the PDE about a pole solution, it was concluded that such pole solutions with a fixed number of poles are asymptotically unstable, while the actual solutions which can be observed experimentally undergo a sequence of kinks between different metastable *groundstates* of the type (7).

To construct the solutions with spawning wrinkles, we exploit the exact correspondence between an arbitrary stochastic dynamical problem with the Gaussian distributed random force and a quantum field theory, [21]. A short and elegant proof had been given in [22]. The stochastic problem (4) and (5) is completely equivalent to the field theory of two fields with an action functional

$$S[u, w] = \frac{1}{2} \int dt dx dy w(x, t) D_f(x - y, t) w(y, t) - i \int dt dx w \left( u_t + U_b u u_x - D_M u_{xx} - \frac{1}{2} \gamma U_b \Lambda \{u\} \right). \quad (11)$$

Here,  $w(x, t)$  is the auxiliary field which comes into play instead of the random force  $f$ .  $w(x, t)$  determines the response functions of the system, for instance, the linear response function is  $\langle uw \rangle$ .

### 3 The power law of the flame front expansion

The mean squared distance of propagating flame front,  $R^2(t)$ , can be expressed naturally via the linear response function mentioned at the end of the previous section as following,

$$R^2(t) = \int dx x^2 \langle u(x, t) w(x, 0) \rangle. \quad (12)$$

The requirement that each term of the action functional be dimensionless (with respect to  $x$  and  $t$  separately) leads to the power counting relation for the product,  $\Delta[uw] = d$ , where  $d$  is the space dimensionality. Therefore, the power of the linear response function is

$$\Delta[\langle uw \rangle] = d - d = 0. \quad (13)$$

We note that (13) is still valid whether a critical regime is attained or not.

Following a tradition, we accept the natural normalization condition that  $\Delta[x] = -1$ , then

$$\Delta[R^2] = -2. \quad (14)$$

From the other hand, the observation data show that

$$R^2 \propto t^3 \quad (15)$$

(i.e.,  $R \propto t^{3/2}$ ). Comparing (14) and (15), one concludes that, in the theory (11), the time variable  $t$  possesses the effective dimension,

$$\Delta[t] = -2/3, \quad (16)$$

which is equal to the Kolmogorov's critical dimension of time in fully developed turbulence.

If the result (16) were a true critical dimension, then one could claim that there is a critical regime in the wrinkling flame front propagation problem, i.e., for any correlation function

there is a definite stable large scale long time asymptotics. Nevertheless, we stress the dramatic difference between the stochastic theory of turbulence (the Navier-Stocks equation with a random forcing included, [18]) and the actual problem. From the point of view of the critical phenomena theory, the problem (4) - (5) is formulated erroneously.

Namely, interesting in the long time large scale asymptotics behavior of correlation functions, one has to omit the term  $\propto D_M k^2$  (in the momentum Fourier space) from the action (11) in benefit to  $\propto \gamma U_b |k|$ . However, in this case, there are infinitely many Green's functions which have singularities with respect to a general dilatation of variables, i.e., such a theory cannot be renormalized.

One can investigate a theory in which the both terms are included simultaneously. Up to our knowledge, a model where the concurrence between two terms (in the momentum Fourier space)  $\propto k^2$  and  $\propto |k|^{2-2\alpha}$  ( $0 < \alpha < 1/2$ ) was considered firstly in [24] in the framework of the renormalization group approach. It is shown that up to the value  $\alpha_c < 1/2$  a regular expansion in  $\alpha$  and  $\varepsilon$  (the deviation of the space dimensionality from its logarithmical value) can be constructed and then summed over by the standard renormalization group procedure. The critical indices of all quantities are still fixed on their kolmogorovian values.

However, for  $\alpha > 1/2$  and for the particular case of  $\alpha_r = 1/2$  which we are interested in, the renormalization group method fails. The matter is in new additional singularities which spawning in the correlation functions of the field  $u$  as  $|k| \rightarrow 0$ . Such singularities cannot be handled by the renormalization group in principle since they do not related to a general scaling with respect to dilatation of variables.

The summation of the leading infrared ( $|k| \rightarrow 0$ ) singularities of correlation functions can be done by an infrared perturbation theory. If we limit ourselves to the functions  $u_z(x, t)$  which have poles  $\{z(t)\}$  in the complex plane, then the action of the singular operator  $\Lambda$  is reduced to a first order derivative operator

$$\Lambda\{u_z\} = i \operatorname{sign}(\Im[z(t)]) \partial_x u_z(x, t), \quad (17)$$

see [11]. Then, in the momentum-frequency representation, the term with  $\Lambda$  can be taken into account as a small shift of frequency ( $\propto \gamma$ ),

$$\omega \rightarrow \omega - i\gamma U_b |k|. \quad (18)$$

The infrared perturbation theory results from the expansion of  $\exp S$  over nonlinearities. The corresponding diagram technique coincide with the diagram technique of Wyld, [25]. The lines in the diagrams are associated with the bare propagators, in the Fourier space,

$$G_{uw} = G_{wu}^* = \frac{1}{-i(\omega - i\gamma U_b |k|) + D_M k^2} \quad (19)$$

and

$$G_{uu} = \frac{1}{-i(\omega - i\gamma U_b |k|) + D_M k^2} D_f(k) \frac{1}{i(\omega - i\gamma U_b |k|) + D_M k^2}, \quad (20)$$

where  $D_f(k)$  is the momentum representation of (6). For any correlation function, this diagram technique gives an infrared representation which is naturally consistent with the  $\gamma$  expansion and is well defined for small values of the parameter  $\gamma$ .

We are not going to discuss the application of the infrared perturbation theory to the actual problem in detail. Here, we conclude that for any pseudo-differential operator  $\propto |k|^{2-2\alpha}$ ,  $0 < \alpha < 1/2$ , the model of the type (11) has a critical regime with the critical indices fixed at their Kolmogorov's values (see [24]). However, for  $\alpha \geq 1/2$  the stability of asymptotics still an important open question if  $\gamma$  is large.

## 4 Statistics of planar flame front wrinkling

We are going to discuss the saddle point configurations of (11) which can provide us by a detailed description of mechanism of wrinkles generation on the propagating flame front surface. For the future purposes, it would be convenient to perform consequently the rescaling of fields in (11),

$$u \rightarrow \frac{u}{U_b}, \quad w \rightarrow U_b w, \quad (21)$$

such that the parameter  $U_b$  is removed from the nonlinear term  $wuu_x$ , and then another rescaling,

$$u \rightarrow \frac{u}{\gamma}, \quad U_b \rightarrow \frac{U_b}{\gamma}. \quad (22)$$

As a result of such a simple transformation, we observe that the parameter of thermal expansion  $\gamma$  which we assume to be small, plays the formal role of  $\hbar$  in quantum field theory:

$$S \rightarrow \frac{1}{\gamma} S. \quad (23)$$

The correlation functions of the basic field  $u$  are then given by the functional integral

$$G_n(x_1, t_1; x_2, t_2; \dots x_n, t_n) = \int \mathcal{D}u \mathcal{D}w u(x_1, t_1) u(x_2, t_2) \dots u(x_n, t_n) \exp(-\frac{1}{\gamma} S), \quad (24)$$

and can be derived naturally by means of a generating functional which has been introduced firstly in [23] and then employed in [26, 27]

$$\mathcal{Z}(\lambda) \equiv \left\langle \exp \left( i \int dt dx \lambda u \right) \right\rangle = \int \mathcal{D}u \mathcal{D}w \exp \left( \frac{1}{\gamma} \{ -S + i \int dt dx \lambda u \} \right). \quad (25)$$

The coefficients of the expansion of  $\mathcal{Z}$  in  $\lambda$  are the correlation functions (24).

There are no general methods to compute such a functional integral exactly. The straightforward perturbative approach is to expand the exponential in the functional integral (25) in powers of the nonlinear term  $wuu_x$ . However, since we are interested in nonperturbative effects, it seems more natural to search for some saddle-point configurations which minimize the action functional (11), thus dominating the functional integral in a way similar to the saddle point

approximation in ordinary integrals. Such solutions are called *instantons*, and they determine the asymptotics of (25) at small  $\gamma \ll 1$  which corresponds to WKB approximation in quantum field theory ( $\hbar \ll 1$ ).

Another quantity which can be expressed via the generating functional (25) is the probability distribution function  $\mathcal{P}(u)$  for the field  $u$ ,

$$\mathcal{P}(u) = \int \mathcal{D}\lambda \mathcal{Z}(\lambda) \exp \left( -i \int dt dx \lambda u \right). \quad (26)$$

The behavior of  $\mathcal{P}(u)$  for large  $u$  is also dominated by some saddle-point configurations of the integrand. However, these configurations are not the same for both (25) and (26).

## 5 Kink solutions of the forced Syvashinsky equation

In what follows we shall look for saddle point configurations driven by the random force in terms of functions which have poles in the complex plane,  $u_z(x, t)$ . To be specific, we observe generation of the four poles configuration from the two poles as a result of a kink. In contrast with [26] and [27], we need not introduce here a large artificial parameter to fix the saddle points dominating the functional integrals (25) and (26) since we have the inverse thermal expansion coefficient  $1/\gamma$  which is naturally large.

Suggesting that the field  $u$  can be continued analytically on the complex plane except for the poles, we shall study the correlation function of the form

$$G(z) = \left\langle \exp \left[ \frac{u(z) - u(z^*)}{\gamma} \right] \right\rangle \quad (27)$$

of two distinct points of the complex plane symmetrical with respect to the real axis. We suppose also that at the initial moment of time  $t = 0$  the field  $u$  can be depicted as a configuration of two complex conjugated poles,  $z$  and  $z^*$ . The function (27) possesses a generating property: Tailoring (27) in powers of  $1/\gamma$ , one obtains the "structure functions" for the field  $u$ . The functional Fourier transform (26) of (27) gives us the two point probability distribution. The structure function generated by (27) is related to the same point  $x = \Re(z)$  on the real axis.

Taking an average in (27) with respect to a functional measure, we perform an integration over all possible configurations  $u(x, t)$  with the asymptote prescribed by initial two poles and all possible final multipole configurations. The basic symmetry of the action (11) is the Galilean invariance which reveals itself in the real transformation

$$u_a(x, t) \longmapsto u(x + X_a(t), t) - a(t), \quad (28)$$

where  $a(t)$  is an arbitrary function of  $t$  decreasing rapidly as  $|t| \rightarrow \infty$ , and  $X_a(t) = \int_0^t dt' a(t')$ . The transformation (28) defines an orbit in the functional space of  $u$  along which the result of functional averaging does not change. It follows that the integral itself is proportional to the

volume of this orbit. This volume should be factorized before one can perform the saddle-point calculation (see [28]). It is appropriate to choose for the latter the "plane" transversal to the real axis  $\Re(z)$  and then cancel out the real components of  $u$  which are related to each other via (28). In (27), the real contribution to  $u$  is subtracted out, so that it is very suitable for instanton calculations.

The asymptotics of small  $\gamma$  in (27) is dominated by the saddle point configurations of the functional

$$\mathcal{W}[u, w, z] = \frac{u(z) - u(z^*)}{\gamma} - S[u, w], \quad (29)$$

which should satisfy the following equations obtained by varying of (29) with respect to  $u$  and  $w$ :

$$u_t + uu_x - D_M u_{xx} - \frac{1}{2} U_b \Lambda\{u\} = -\frac{1}{2} \frac{i U_b}{\gamma} \int dx' D_f(x - x', t) w(y, t), \quad (30)$$

$$w_t + ww_x + D_M w_{xx} + \frac{1}{2} U_b \Lambda\{w\} = -\frac{i}{\gamma} \delta(t) \{\delta(x - z) - \delta(x - z^*)\}. \quad (31)$$

These equations for the saddle point configurations are similar to those derived in [26] except the last singular term in the l.h.s. They follow from the Syvashinsky equation for the slope function (4), however they contain information on a special force configuration necessary to produce instantons also.

Indeed, the particular solutions of (30) and (31) are dependent substantially from the initial data for  $u$  and  $w$ . Minimization of the action requires  $u \rightarrow 0$ , at  $t \rightarrow -\infty$  and  $w \rightarrow 0$ , at  $t \rightarrow \infty$ . Obviously, any solution of the equation (31) which is nonsingular as  $t \rightarrow +\infty$  should be equal to zero at  $t > 0$  (since the field  $w$  feels a negative diffusivity). Following an analogy with [26, 27], one can say that the field  $w$  propagates backwards in time starting from its initial value

$$w(t = -0) = -\frac{i}{\gamma_0} \{\delta(x - z(0)) - \delta(x - z^*(0))\} \quad (32)$$

while it is zero at all later moments of time. Therefore, the system (30) and (31) as well as the integrals in (11) can be treated for  $t < 0$  only.

While propagating backward in time, (32) is a subject to a drift of the initial conditions as governed by the velocity in the equation (31), the smearing of the initial  $\delta$ -function distributions in (32) due to diffusivity, and finally, an advection in the complex plane, in the imaginary direction towards the real axis. In the limit of no diffusivity, one can neglect the smearing in the equation (31). A simplified equation, which we arrive at when we drop the diffusivity term is just moving the  $\delta$ -singular right hand side of (31) around. Therefore, the solution of (31) can be expressed naturally in the form

$$w(t) = -\frac{i}{\gamma(t)} \{\delta(x - z(t)) - \delta(x - z^*(t))\} \quad (33)$$

with the boundary conditions  $\gamma(0) = \gamma_0$ ,  $z(0) = z_0$ . If one takes the diffusivity term in (31) into account (in the case of eventually small  $D_M/\gamma U_b \leq 1$ ), the solution of (31) would be

expressed in terms of even functions decaying as  $|x| \rightarrow \infty$ ,

$$w(t) = -\frac{i}{\pi\gamma(t)} \left\{ \frac{y(t)}{y^2(t) + (x - z(t))^2} - \frac{y(t)}{y^2(t) + (x - z^*(t))^2} \right\}, \quad (34)$$

where the parameter

$$y(t) = \frac{D_M}{\gamma(t)U_b} \quad (35)$$

is a size of an instanton, and  $z(t)$  is its position changing with time (we have borrowed the terminology from the quantum field theory). As  $|y(t)| \rightarrow 0$ , the instanton shrinks to a point, and the solution (34) is reduced to (33).

To proceed with the equation (30), we rewrite it due to (17) in the form

$$u_t + uu_x - D_M u_{xx} - \frac{i}{2} U_b \text{sign} \Im[z(t)] \partial_x u_z(x, t) = -\frac{1}{2} \frac{iU_b}{\gamma} \int dx' D_f(x - x', t) w(y, t). \quad (36)$$

The one-dimensional equation (36) can be linearised by the Cole-Hopf transformation,

$$u(x, t) = -2D_M \frac{\psi_x(x, t)}{\psi(x, t)}, \quad (37)$$

so that we arrive at the equation

$$\psi_t - D_M \psi_{xx} - \frac{i}{2} U_b \text{sign} \Im[z(t)] \partial_x \psi(x, t) = -\frac{1}{2} \frac{iU_b}{\gamma_0} \int dy D_f(x - y, t) w(y, t). \quad (38)$$

Since  $w(x, t)$  distinguishes from zero only for  $t \leq 0$  and  $D_f \propto \delta(t)$ , the only nontrivial contribution into the r.h.s. of (38) is given by the moment of time  $t = 0$ . The general solution of the equation (38) in the Fourier space reads as follows

$$\psi_{inst}(k, t) = \frac{1}{2} \frac{iU_b}{\gamma_0} D_f(k, 0) w(k, 0) \delta(t), \quad (39)$$

plus a transient process decaying rapidly as time growing,  $\propto \exp[-t(D_M k^2 + U_b |k|)]$ .

Now we can use (6) and (34) to write down the r.h.s. of (39) explicitly,

$$D_f(k, 0) w(k, 0) = 4\pi^2 i D_0 \exp[-(y(0) + m)|k| - ik\Re[z_0]] \sin k\Im[z_0]. \quad (40)$$

Performing an inverse Fourier transform of (39), one obtains

$$\psi_{inst} = \frac{D_0 U_b^3 \gamma_0}{2\pi} \left[ \frac{1}{(D_M + \gamma_0 U_b m)^2 + \gamma_0^2 U_b^2 (x - z_0)^2} - \frac{1}{(D_M + \gamma_0 U_b m)^2 + \gamma_0^2 U_b^2 (x - z_0^*)^2} \right]. \quad (41)$$

Finally, we arrive ([29]) at the following four poles configuration for the instanton  $u_{inst}$ ,

$$u_{inst} = \frac{4D_M x \Im[z]}{((D_M/\gamma_0 U_b + m)^2 + x^2 - \Im[z]^2)^2 + 4x^2 \Im[z]^2} \delta(t). \quad (42)$$

For the last step of instanton computation we have to define the functions  $\gamma(t)$  and  $\varphi(t) \equiv \Im[z(t)]$  in the equation (34) (remember that  $\Re[z]$  is fixed) as  $t < 0$ . We can do it by a direct

substitution of (33) (in case of eventually small diffusivity) into the equation (31). Here we note that since the instanton solution (42) exists as  $t \geq 0$  (if one takes the transient process into account), we have to use the initial two pole configuration instead of  $u(x, t)$  in (31). As a result, we obtain the system of simplified equations

$$-\dot{\gamma}(t) = \frac{4xD_M}{x^2 + \varphi(t)^2} \gamma(t), \quad (43)$$

$$-\dot{\varphi}(t) = \frac{4xD_M}{x^2 + \varphi(t)^2} \varphi(t). \quad (44)$$

The formal solution of (43) is given by

$$\gamma(t) = \gamma_0 \exp \left( - \int_t^0 \frac{4xD_M}{x^2 + \varphi(t')^2} dt' \right), \quad (t' < 0) \quad (45)$$

and can be computed, in principle, if one knows  $\varphi(t)$ . The equation (44) is equivalent to

$$\frac{x}{D_M} \ln \varphi(t) + \frac{1}{2xD_M} \varphi^2(t) + t = C, \quad (46)$$

which leads to

$$\varphi(t) = \varphi_0 \exp \left[ -\frac{tD_M}{x} - \frac{1}{2} W \left( \frac{e^{-2tD_M/x}}{x^2} \right) \right]. \quad (47)$$

$W(x)$  is the Lambert function which meets the equation

$$W(x) \exp W(x) = x. \quad (48)$$

The latter equation has an infinite number of solutions for each (non-zero) value of  $x$ .  $W$  has an infinite number of branches numbered by an integer number  $n \in [-\infty \dots \infty]$ . Exactly one of these branches is analytic at 0 (the principal branch,  $n = 0$ ). The other branches all have a branch point at 0. The principal branch is real-valued for  $x$  in the range  $-\exp(-1) \dots \infty$ , while the image of  $-\infty \dots -\exp(-1)$  under  $W(x)$  is the curve  $-y \cot(y) + yi$ , for  $y \in [0 \dots \pi]$ . For all the branches other than the principal branch, the branch cut dividing them is the negative real axis. The image of the negative real axis under the branch  $W(n, x)$  is the curve  $-y \cot(y) + yi$ , for  $y \in [2k\pi \dots (2k+1)\pi]$  if  $k > 0$  and  $y \in [(2k+1)\pi \dots (2k+2)\pi]$  if  $k < -1$ . These curves, therefore, bound the ranges of the branches of  $W$ , and in each case, the upper boundary of the region is included in the range of the corresponding branch.

Each particular orbit of (47) provides a distinct solution of (45) and (33). However, each configuration  $w(x, t)$ ,  $t < 0$  which enjoys (33) is related to the same configuration  $u(x, t)$ ,  $t > 0$ . The value of (29) for the instanton is obviously finite, however, one can hardly compute it for each branch of  $w(z, t)$ .

Let us consider the principle branch of the function  $\varphi(t)$  just to illustrate the idea of computation. One can check that the leading contribution to  $\varphi$  is accumulated around  $x = 0$ . The asymptotic behavior of  $W$  at complex infinity and at 0 is given by

$$W(x) \sim \log(x) - \log(\log(x)) + \sum_{m,n=0}^{\infty} C(m, n) \frac{\log(\log(x))^{(m+1)}}{\log(x)^{(m+n+1)}}, \quad (49)$$

where  $\log(x)$  denotes the principal branch of the logarithm, and the coefficients  $C(m, n)$  are constants.

Restricting to the first term of the asymptote (49), one obtains

$$\varphi(t) \simeq -\varphi(0)D_M t, \quad (t < 0), \quad (50)$$

then we use (50) and (45) to compute  $\gamma(t)$  as  $t < 0$ ,

$$\gamma \simeq \gamma_0 \exp\left[\frac{4}{\varphi_0} \tan^{-1} \frac{\varphi_0 D_M t}{x}\right], \quad t < 0. \quad (51)$$

Now it is a matter of a simple computation to find the action on the principal branch of the instanton. We collect everything together and substitute (51), (50), (42), and (33) back to (11) to obtain

$$S_{inst} = -\frac{D_0}{m} \exp\left[-\frac{2}{\varphi_0} \tan^{-1} \varphi_0\right], \quad (52)$$

while the correlation function we have been studying is

$$G \propto \exp\left(\frac{D_0}{m} \exp\left[-\frac{2}{\varphi_0} \tan^{-1} \varphi_0\right]\right). \quad (53)$$

This simplified formula is obviously not an exact answer. It is just a leading asymptotic of  $G(\Im[z_0])$  if  $1/\gamma_0$  is a large number,  $\varphi_0 \equiv \Im[z_0]$  is eventually small, however, not too small (since we have not taken the smearing due to diffusivity into account). Nevertheless, for some interval of scales  $\Im[z_0]$ , rather small than large (see Fig. 1), the asymptotics prescribed by (53) (a thick line) is very close to a model curve which corresponds to the Kolmogorov's critical exponent for velocity,  $\Delta^K[v] = -1/3$ .

Due to a specific property of the action (11), the contribution from fluctuations of  $\delta w$  and  $\delta u$  up to second order will be zero. To find higher order corrections to (53), one has to consider at least the third order terms. We investigate this problem in future publications.

## 6 Discussion and Conclusions

The study performed in this paper confirms that the stochastic model for the outward propagating flame in the regime of well developed hydrodynamic instability leads to the self-fractalization of the flame front. The mechanism consists of successive instabilities through which the interface becomes more and more wrinkled as time increases. The main contribution to the statistics of wrinkles is due to a coupled field-force configurations which are found to be responsible for the birth and growth of wrinkles. This behavior is different from exact pole solutions for which the number of poles is constant. The acceleration of the mean front radius is clearly due to successive births of poles.

We have demonstrated that the  $t^{3/2}$  power law observed experimentally for the mean radius of the advancing flame front would be a direct consequence of the Kolmogorov's scaling with the

critical dimensions of time  $\Delta_t^K = -2/3$  which is well known in the fully developed turbulence theory. Provided the critical regime in the model of wrinkling flame front exists, then it means that each correlation function in the model has a definite stable long time large scale asymptotics, which does not depend from the particular sequence of kinks. If one replaces the nonlocal operator  $\Lambda$  with some pseudo-differential operator  $\propto |k|^{2-2\alpha}$ ,  $0 < \alpha < 1/2$ , then, as it was shown in [24], the model of the type (11) has a critical regime with the critical indices fixed at their Kolmogorov's values. However, for  $\alpha \geq 1/2$  the stability of asymptotics still an important open question if  $\gamma$  is large.

We have used the saddle point calculations assuming the inverse thermal expansion coefficient  $1/\gamma$  as a large parameter. As a result, we construct an infinite family of instanton solutions numbered by  $n \in [-\infty \dots \infty]$ . Each instanton determined by one of the branches of the Lambert function  $W(x)$  has a unique behavior as  $t < 0$ , however, all instantons are undistinguishable as  $t > 0$ . The asymptotic behavior of these solutions is close to that of prescribed by the Kolmogorov's scaling.

The crucial problem of the developed technique is of contribution to the action from the fluctuations against the instanton background. The general analysis of the set of instantons which we have found will be published in a forthcoming paper.

## 7 Acknowledgments

We would like to thank E. Khanin, M. Nalimov, and M. Komarova for their valuable advices and discussions.

## References

- [1] L. Filyand, G.I. Sivashinsky, and M.L. Frankel, *Physica D* **72**, 110 (1994).
- [2] L.D. Landau, *Acta Phys. Chem. USSR* **19**, 77 (1944).
- [3] G.Darrieus (unpublished).
- [4] D. M. Michelson, G.I. Sivashinsky, *Acta Astronautica* **4**, 1207 (1977).
- [5] O. Rahibe, N. Aubry, G.I. Sivashinsky. *Phys. Rev. E*, **54**, 4958 (1996).
- [6] O. Rahibe, N. Aubry, G.I. Sivashinsky. *Combust. Theory Modelling*, **2**, 19 (1998).
- [7] O. Rahibe, N. Aubry, G.I. Sivashinsky, and R. Lima, *Phys. Rev. E* **52**, 3675 (1995).
- [8] Ya. B.Zel'dovich, B. Istratov, A. G. Kidin, and V.B. Librovich, *Combust. Sci. Technol.* **24**, 1 (1980).

- [9] P. Pelcé, *Dynamics of Curved Fronts* (Academic, New York, 1989).
- [10] P. Pelcé, P. Clavin, Europhys. Lett. **3**, 907 (1987).
- [11] O. Thual, U. Frish, and M. Henon, J. Phys. (France) **46**, 1485 (1985).
- [12] G. Joulin and P. Cambray, Combust. Sci. Tech. **81**, 243 (1992).
- [13] M. Renardy, Physica D **28**, 155 (1987).
- [14] D. Vaynblat, M. Matalon. Preprint: Stability of Pole Solutions for planar propagating flames (Parts I and II), Submitted for publication (1998).
- [15] G. Joulin, in *Combustion, Detonation, and Shock Waves*, edited by S.M. Frolov (Combustion Institute, Moscow ), Vol. 1 (1996).
- [16] O. Kupervasser, Z. Olami, I. Procaccia, Phys. Rev. Lett., 76, 1, 146 (1996).
- [17] L'vov V.S., Phys. Rep., v. 207, 1 (1991).
- [18] Adzhemyan L.Ts., Antonov N.V., Vasiliev A.N., *Field Theoretic Renormalization Group in Fully Developed Turbulence*, Gordon and Breach Science Publ., (1998).
- [19] Adjemyan L. Ts., Vasil'ev A. N., Antonov N. V., Phys. Rev. E, **58**, 2, 1823 (1998).
- [20] Kraichnan R.H., Phys. Fluids, v. 7, 1723 (1964), v.8, 575 (1965), v. 9, 1728, 1884 (1966).
- [21] P.C. Martin, E.D. Siggia, H.A. Rose, Phys. Rev. A8, 423 (1973).
- [22] Adjemyan L. Ts., Vasil'ev A. N., Pis'mak Yu. M., Theor. and Math. Phys. **57**, 1131 (1983).
- [23] C. de Dominicis, J. Phys. (Paris) Colloq. **37**, 201-247 (1976).
- [24] Nalimov M. Yu., Udalov A.A., Vestnik SPbGU (Acta of Saint-Petersburg State University), **4**, 4 (25), p.33 (1996). (in Russian)
- [25] H.W. Wyld, Ann. Phys. **14**, 143 (1961).
- [26] V. Gurarie, A. Migdal, Phys. Rev. E, **54**, 5, 4908 (1996).
- [27] G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E, **54**, 5, 4896 (1996).
- [28] Faddeev L.D., Popov V.N. Physics Lett. 25 B (1), 29 (1967).
- [29] To compute the inverse Fourier transform, we have used the fact that for any  $\xi > 0$ ,

$$e^{-\xi} = \int_0^\infty \frac{e^{-\mu}}{\sqrt{\pi\mu}} e^{-\xi^2/4\mu} d\mu.$$

